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# GENERIC SMOOTH MAPS OF SURFACES

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A generic map of a smooth surface $M$ to an oriented smooth surface $N$ is an immersion except on a compact family of curves where it may have fold or cusp singularities. If the domain is oriented, the map is homotopic to one having no cusps while, if nonorientable, it is homotopic to a map having at most one or no cusps depending upon whether the genus of $M$ is odd or even. Conditions under which a generic map is the composition of an immersion of $M$ into $N\times R$ followed by projection to $N$ are given. Finally, an immersion of $P^2$ into $R^3$, whose projection to $R^2$ has a fold locus consisting of a single curve containing a single cusp, is described. The purpose of this paper is to describe a slight improvement upon a theorem of Levine concerning the elimination of cusps of a generic map from one compact oriented surface to another and to elaborate upon results of Haefliger concerning the factorization of maps of surfaces into the plane as the composition of an immersion followed by a projection.

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generic smooth map, singularity, fold, immersion, cusp

## 1. Introduction and statement of results

Whitney [8], in his study of smooth mappings of the plane into the plane, has shown that for any such mapping there is an arbitrarily small smooth homotopy to a smooth map having a particularly simple behaviour. Such a map is called a *generic map* because nearby maps have the same structure, i.e., they form an open dense subset of the space of maps. For such maps either the rank of the Jacobian is two at a point (where the map is a *local diffeomorphism*), or has rank one (the set of such points consists of smooth disjoint curves called *folds*), or has rank zero on the fold curves (the set of such points is a discrete set called the *cusp points*). Since the Whitney result is entirely a local result it is possible to show that for any map from one compact surface to another there is an arbitrarily small smooth homotopy to a smooth generic map.

Suppose that $M$ is a compact connected surface, that $N$ is an oriented connected surface and that $f$ is a generic map of $M$ into $N$. Levine [4] has proved that, if $M$

is oriented, $f$ is homotopic to a generic $g$ such that on each fold curve of $g$ there are at most two cusps. We shall prove the following improvement and extension in this paper.

**Theorem 1.** (1) *If $M$ is oriented then $f$ is homotopic to a generic $g$ having no cusps.*

(2) *If $M$ is non-orientable then $f$ is homotopic to a generic $g$ having at most one cusp or no cusps depending upon whether the genus of $M$ is odd or even.*

The method of proof is a modest extension of that employed by Levine. The development of these ideas will allow us to illuminate and extend results of Haefliger [2] concerning the factorization of maps into immersions followed by projections. Specifically, given a generic map $f: M \to N$ when does there exist an immersion

$$\begin{tikzcd} \bar{f}: M \arrow[r, dashed] \arrow[dr, "f"'] & N \times R \arrow[d, "\pi"] \\ & N \end{tikzcd}$$

so that $f = \pi \circ \bar{f}$, where $\pi$ denotes the projection onto the first factor? Haefliger proves that a generic map $f$ of a compact surface into the plane $R^2$ can be factored by an immersion, $\bar{f}$, into $R^3$ if and only if on each fold curve of $f$ the number of cusps is even or odd depending upon whether the fold curve has a cylinder or Möbius neighborhood. An extension of Haefliger's ideas will allow us to prove the following theorem.

**Theorem 2.** *A generic map $f$ of a compact surface $M$ into a compact surface $N$ can be factor through an immersion $\bar{f}$ in $N \times R$ if and only if on each fold curve of $f$ the number of cusps is even or odd depending upon whether the fold curve has a cylinder or Möbius neighborhood.*

Thus we see that the answer does not depend upon the orientability of $N$. In a concluding section is a discussion of Haefliger's example of a map of a sphere into $R^2$ which cannot be factored through an immersion. In the context of the previously developed method used to prove Theorem 1 this allows us to show that the geometric obstructions to factoring a fixed map through an immersion arise or, if one wishes, can be removed in a very simple way.

There is also a description of a particularly attractive immersion of the projective plane into $R^3$ whose projection to $R^2$ has a single fold curve and a single cusp point. As shown by Whitney [8, Theorem 30A], this is the simplest configuration possible. The most common projections of specific immersions of $RP^2$ seem to have disconnected fold curves or three cusps as, for example, with Boy's surface as pictured by Hilbert and Cohn-Vossen. Other loci actually arise from map images rather than immersions as they must include pinch points. N. Kuiper showed me a series of such pictures dating from a 1960 response to Haefliger's paper [2] which included a description of the locus and two pinch points which, by Haefliger's theorem, could have been removed to achieve an immersion. Conversations with a variety of colleagues who enjoy such problems leads me to conjecture that this must be the simplest and most beautiful fold locus of a generic projection of a projective plane immersed in $R^3$. An enjoyable way of encountering the immersion is to begin with a three cusp projection of Boy's surface and cancel two of the cusps against each other. One should also note the remarkable connection between this locus and the immersed circle bounding two distinct immersed disks attributed to Milnor, Poenaru [5].

## 2. Methods of modification and computation

The methods of Thom [1, 6, 7] allow us to relate the geometric structure of the folds and cusps to the algebraic topology of the surface. Thus we shall need to develop some concepts, vocabulary, and graphical notation to enable us to describe the relations and the geometric methods of modifying a generic map. Suppose that $f$ is a generic map from $M$, a compact connected surface, to $N$, a compact connected oriented surface. Let $C_f$ denote the family of fold curves of $f$ and let $D_f$ denote the discrete subset of cusps. If $M$ is oriented $C_f$ separates $M$ into two regions with $C_f$ as common boundary determined by whether $f$ preserves or reverses the orientation. We shall call these the positive and negative regions, respectively.

Although it is not possible to globally realize maps by projections of immersions it is possible to do so locally and this fact will allow us to describe the basic structure in a geometric fashion. For example, when $M$ is oriented we may distinguish between two types of cusps according to whether the cusp 'points from' the positive region or the negative region. We shall call them positive or negative respectively. To do this consider a realization of the map at the cusp point as the graph of a function followed by a projection as shown in Fig. 1. The key observation is that the image of the negative region of the cusp points into the region where the orientation agrees.

There are three useful procedures for describing homotopies between generic maps which change the structure of the folds and cusps. First if one has two cusps pointing into the same component of the complement of $C_f$ one may 'run them together' along a connecting path and thereby cancel them. This process is illustrated in Fig. 2.

The second modification is the introduction of two cusps by means of a *twist.* This process introduces a pair of cusps, one positive and the other negative.

The third modification is an exchange of cusps. This can occur when one has a positive and a negative cusp which appear on the boundary curves of a connected region.

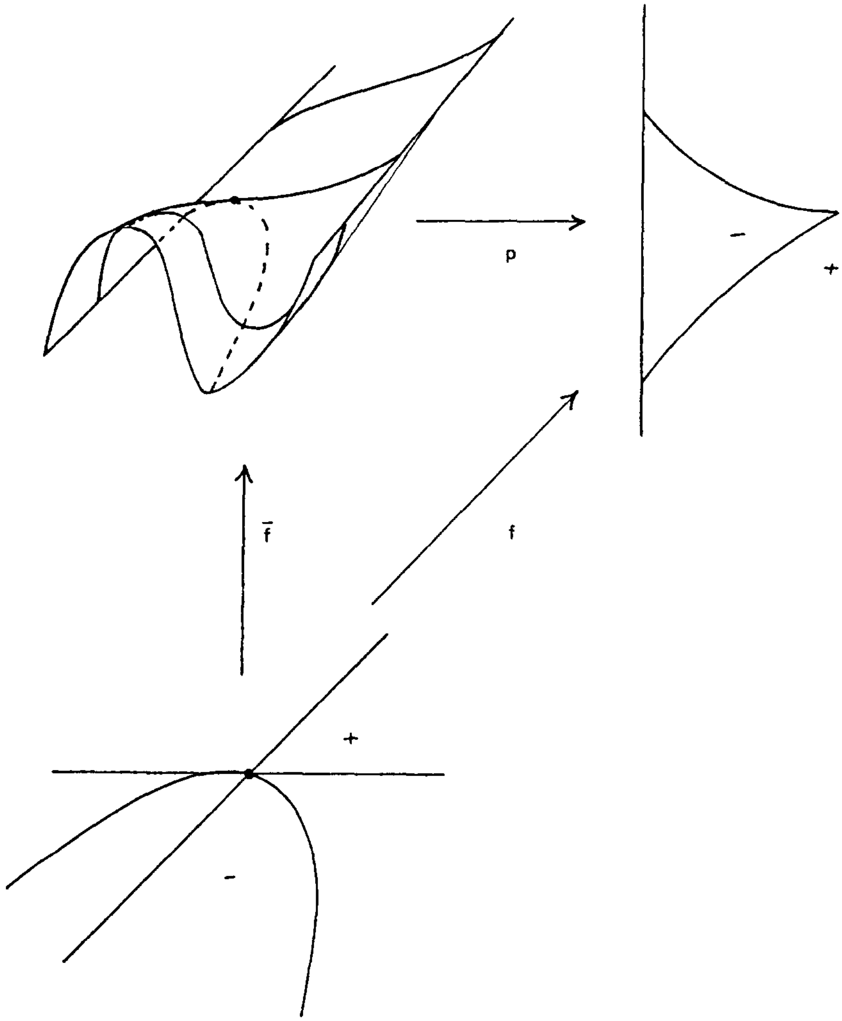


Fig. 1. A negative cusp, $\bar{f}(x, y) = (x, y, xy + x^3)$, $p(x, y, z) = (y, z)$, $f(x, y) = (y, xy + x^3)$.

## 3. Proof of Theorem 1

We shall first consider the case of $M$ and $N$ oriented. In this case the fold curves, $C_f$, determine a homology class $[C_f] \in H_1(M, \mathbb{Z})$ which is trivial since it is the boundary of the region where $f$ preserves the orientation. By introducing cusps via the twist modification and running them together we define a homotopy to a generic map $f_1$ such that $C_{f_1}$ is a simple closed curve in $M$. Since $C_{f_1}$ is the boundary of the positive region there is a cobordism between $C_{f_1}$ and the boundary of a disk which is interior to the positive region. This cobordism has a handle structure which enables one to define a homotopy to a generic map $f_2$ such that $C_{f_2}$ is the boundary of the

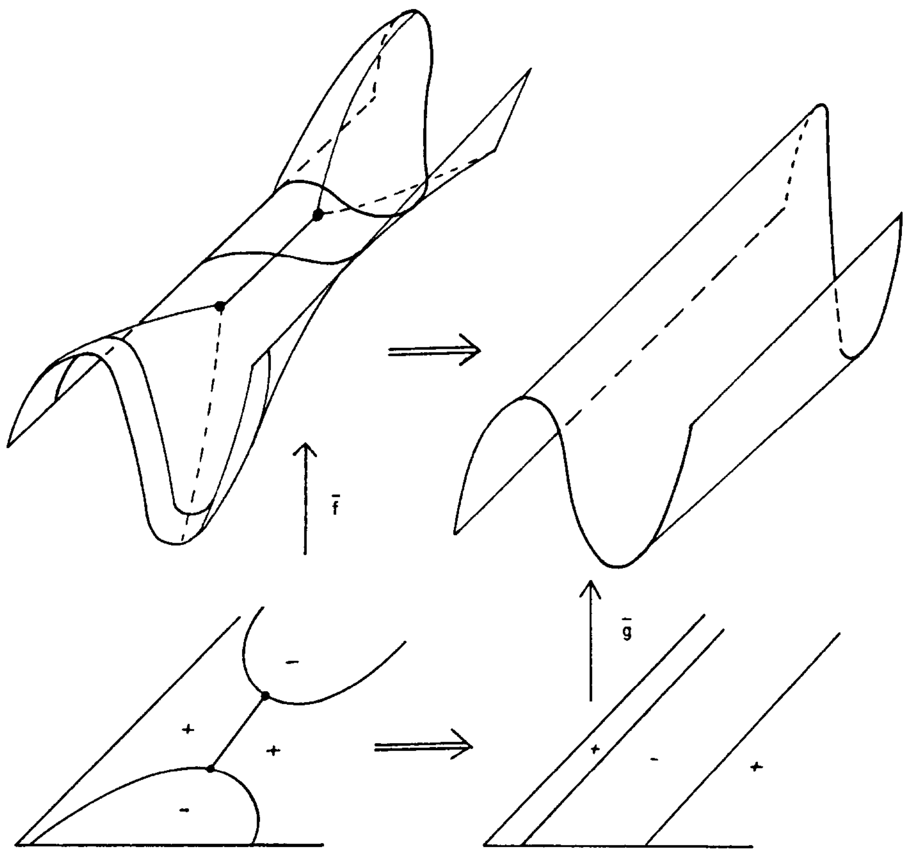


Fig. 2. Running cusps together.

positive region which is now a disk. This is accomplished by using twists to introduce cusps which may be run together following the prescription provided by the level curves of the cobordism and the one handles as they are added. In other words, the positive region associated to $f_2$ may be cut into a disk via cuts along a finite collection of disjoint nonseparating arcs. Cusps may be introduced on the ends of the arcs and then run together along the arcs. This is schematically illustrated in Fig. 5.

The resulting generic map, $f_2$, may have a very complicated structure since we have created a large number of cusps. Nevertheless we may choose a small disk $D$ in the positive component of the complement of $C_{f_2}$ such that $f_2|_D$ is a diffeomorphism and such that $f_2(D) \cap f_2(C_{f_2}) = \emptyset$. Then $f_2^{-1}(f_2(D))$ is a finite collection of smooth disks and, if we define $M_2 = M \setminus (\text{interior}\, f_2^{-1}(f_2(D)))$ and $N_2 = N \setminus (\text{interior}\, f_2(D))$, $f_{2M_2}: M_2 \to N_2$ is a proper generic map between manifolds with boundary. Let $i$ denote the extension of the standard orientation preserving immersion of $N_2$ into $R^2$ to a generic map of $N$ having $2g$ negative cusps where $g(N)$ is the genus of $N$. Such an immersion is pictured in Fig. 6.

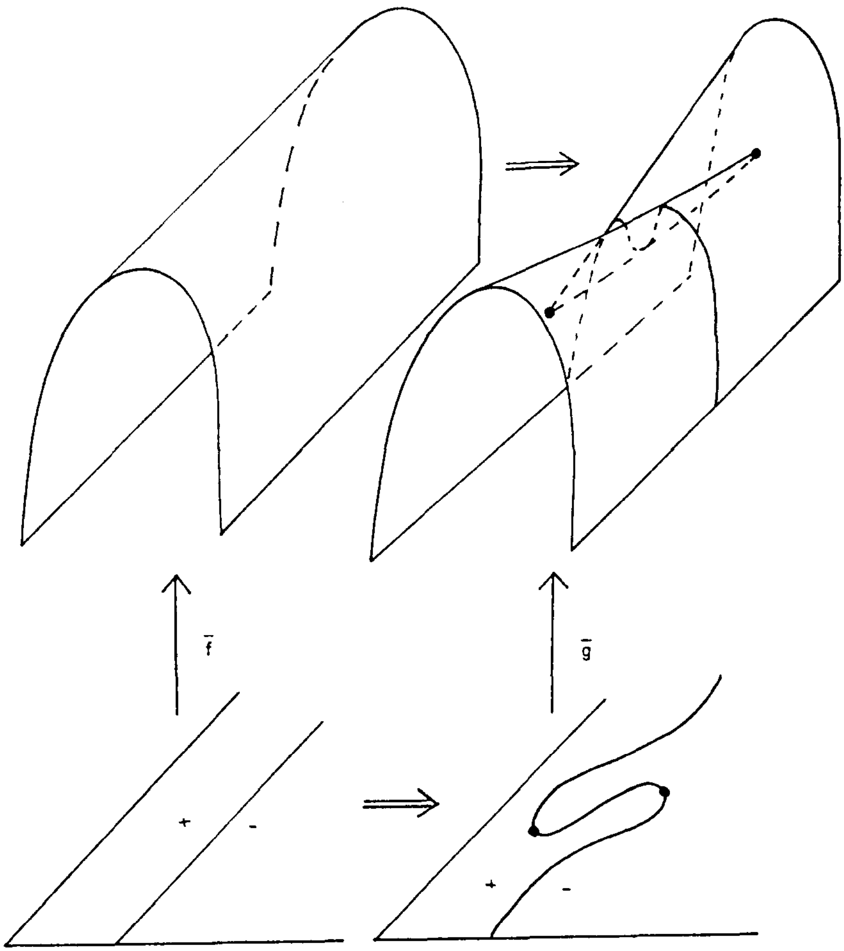


Fig. 3. Twist.

Let $\hat{f}_2 = i \circ f_2$ and let $d_+$ and $d_-$ denote the number of disks of $f_2^{-1}(f_2(D))$ in $M^+$ and $M^-$, the positive and negative components of the complement of $C_{f_2}$. For each of the $d_+$ disks $\hat{f}_2$ introduces a new circle component to $C_{f_2}$, $2g(N)$ negative cusps, and a new disk to the negative region. Similarly, for each of the $d_-$ disks $\hat{f}_2$ introduces a new circle component to $C_{f_2}$, $2g(N)$ positive cusps and a new disk to the positive region.

Let $n_+$ ($\hat{n}_+$) and $n_-$ ($\hat{n}_-$) denote the number of positive and negative cusps of $f_2$ ($\hat{f}_2$). From the above we have the following relations:

$$\hat{n}_+ = n_+ + (2g(N))d_-$$

$$\hat{n}_- = n_- + (2g(N))d_+$$

$$X(\hat{M}^+) = X(M^+) + d_- - d_+$$

$$X(\hat{M}^-) = X(M^-) + d_+ - d_-.$$

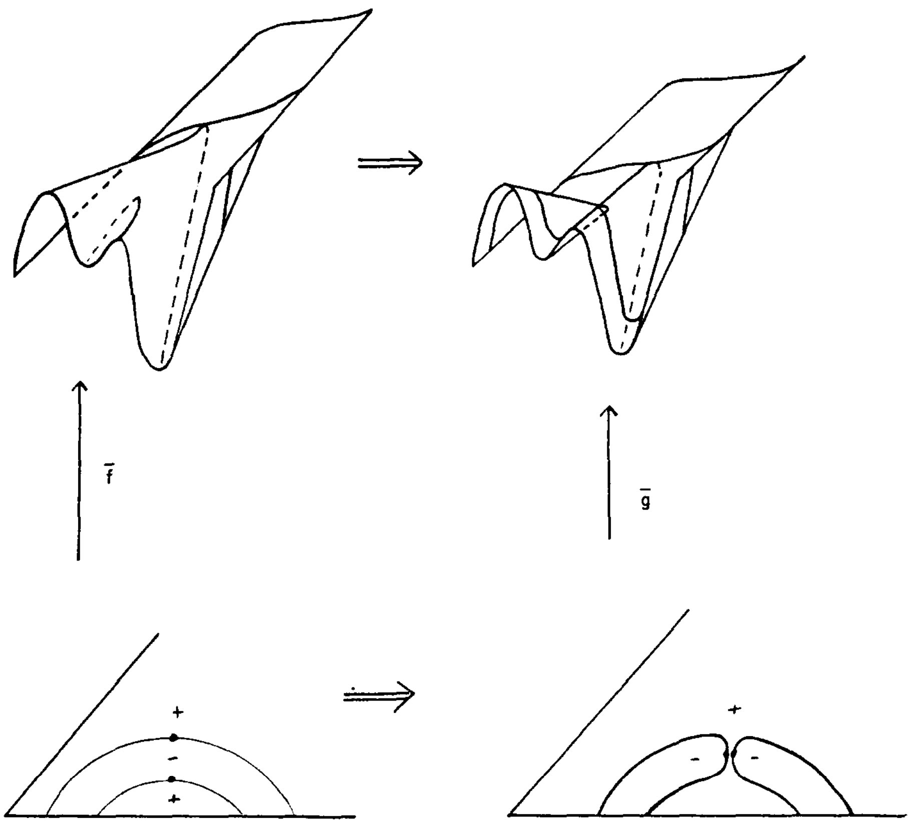


Fig. 4. Cusp exchange.

We shall apply the following observation of Haefliger [2], page 54 to $i \circ f_2 = \hat{f}_2 : M \to R^2$.

**Observation** (Haefliger). $X(\hat{M}^+) - X(\hat{M}^-) = \hat{n}_+ - \hat{n}_-$. *Thus* $X(M^+) - X(M^-) = n_+ - n_- - 2(g(N) - 1)(d_+ - d_-)$.

By our construction $X(M^+) = 1$ and $X(M^-) = 1 - 2g(M)$ so that $n_+ - n_- = 2g(M) + 2(g(N) - 1)(d_+ - d_-)$ which is even. Thus either both $n_+$ or $n_-$ are even or both are odd. In the latter case we introduce a twist which adds a positive and a negative cusp to ensure that both are even for the resulting generic map which we shall denote by $f_3 : M \to N$. As a result of the proof to this stage we have the following generalization of Haefliger's observation.

**Observation.** *If* $f : M \to N$ *is a generic map of degree* $d(f)$ *then* $n_+ - n_- = 2(X(M^+) - (g(M) - 1) + (g(N) - 1)d(f))$.

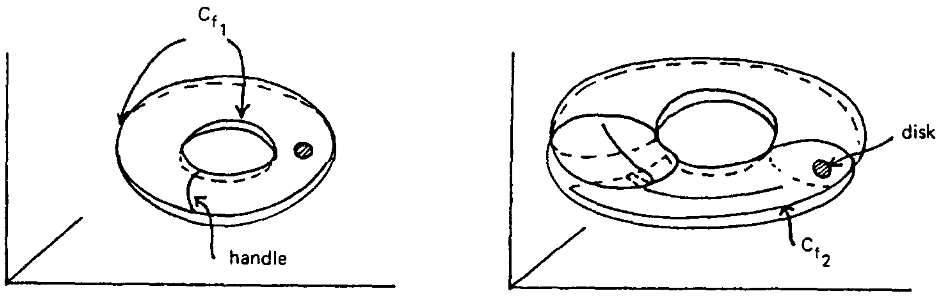


Fig. 5. Adding a handle and cancelling cusps.

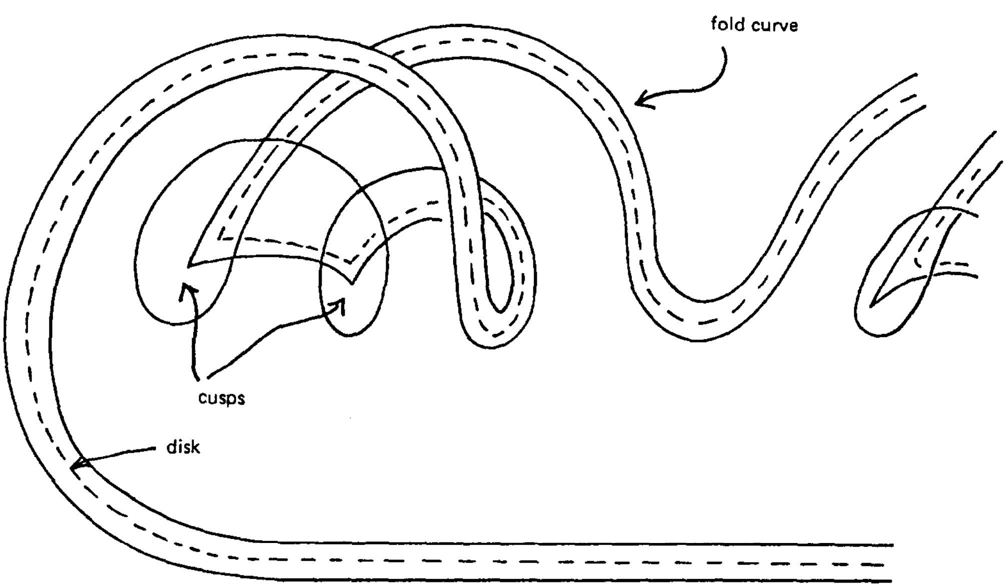


Fig. 6.

To complete the proof for the case where $M$ is oriented we first run adjacent cusps together if they are both positive or negative, for example a locus of the form Fig. 7(i) can be replaced by the locus Fig. 7(ii) so that $f_3$ is homotopic to a generic $f_4$ such that $C_{f_4}$ is the union of curves without cusps and a curve with an even number of positive and negative cusps which alternate.

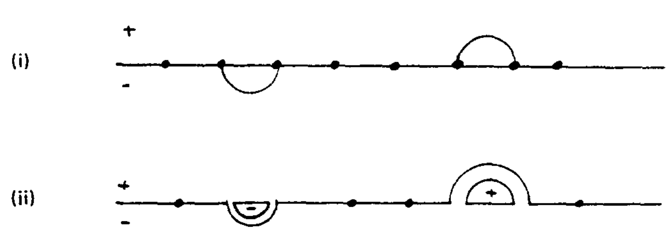


Fig. 7.

By a sequence of pairs of applications of cusp exchanges and running cusps together we find a homotopic generic $g$ without cusps as shown in Fig. 8(i) by first an exchange of cusps, Fig. 8(ii), and then run the cusps together, Fig. 8(iii), to remove the adjacent cusps of opposite signs. Let $g$ denote the resulting generic map. Thus we have proved case (1) of Theorem 1.

The method of proof for the second part of Theorem 1 is quite similar. First we shall employ a result of Thom [1].

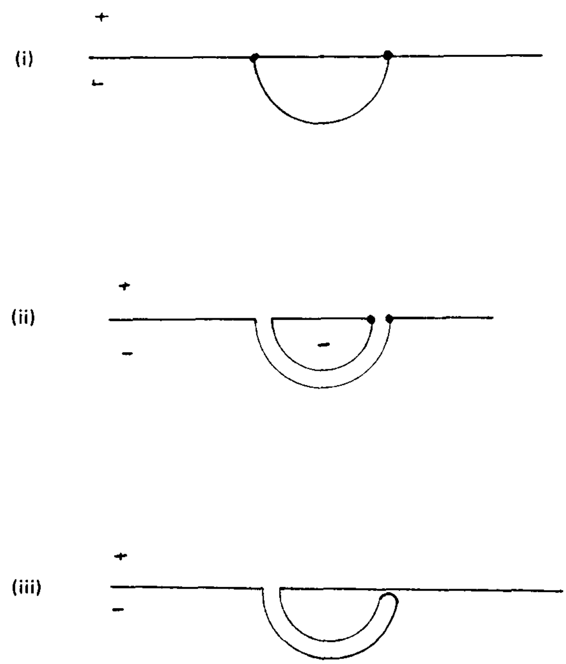


Fig. 8.

**Theorem** (Thom). *If $f$ is a generic map of a closed surface $M$ into an oriented surface $N$, the* mod 2 *cohomology classes dual to $C_f$ and* {cusps *of $f$*} *are the first and second Stiefel–Whitney classes, respectively.*

Since we are assuming that $M$ is non-orientable the first Stiefel–Whitney class is non-zero so that, following a homotopy from $f$ to $f_1$, as in the previous case, $C_{f_1}$ is a simple closed curve and does not separate. $M$ is the connected sum of $g(M)$ real projective planes and has Euler characteristic $X(M)=2-g(M)$. Furthermore the evaluation of the second Stiefel–Whitney class on the fundamental class of $M$, the second Stiefel–Whitney number, is equal to the Euler class modulo 2. Thus the number of cusps is equal to the genus of $M$, $g(M)$, modulo 2.

As in the orientable case we may assume that adjacent cusps are not on the same side as we would run these together. If they are opposite we use the exchange and run the resulting cusps together. Thus as in the orientable case we can eliminate all cusps if $g(M)$ is even and must terminate with one cusp if $g(M)$ is odd thereby proving the second case of Theorem 1. □

## 4. Factoring a generic map by an immersion

Suppose that $f: M \to N$ is a generic map and $p: N \times R \to N$ is the projection of $N \times R$ onto its first factor. Theorem (2) asserts that there is an immersion $\bar{f}: M \to N \times R$ such that $f = p \circ \bar{f}$ if and only if on each fold curve of $f$ the number of cusps is even or odd depending upon whether the fold curve has a cylinder or Möbius neighborhood in $M$, respectively. The proof is a direct extension of the two lemmas employed by Haefliger for the case $N = R^2$.

**Lemma 1.** *The number of* cusps *on a component $C$ of $C_f$ is even if a sufficiently small neighborhood of $C$ and the bundle* ker $df|_C$ *are both orientable or non orientable and odd, if one is orientable and the other is not orientable.*

**Lemma 2.** *A generic map $f$ of $M$ to $N$ can be expressed as $p \circ \bar{f}$ if and only if the bundle* ker $df|_C$ *is orientable.*

The proof of Lemma 1 is exactly as given in Haefliger while that of Lemma 2 requires only the following observation to extend the proof to all surfaces.

Consider the image of a component $C$ of $C_f$. Independently of the number of cusps which appear on $C$ its image has only one side, globally, i.e., the side to which it 'folds' is unchanged by the appearance of a cusp as illustrated in Fig. 9. As a consequence the image of $C$ has a cylinder neighborhood which we may take to be of $R^2$ so as to take advantage of Haefliger's proof in the $R^2$ case to construct a height function for an immersion in a neighborhood of $C$ which projects to the

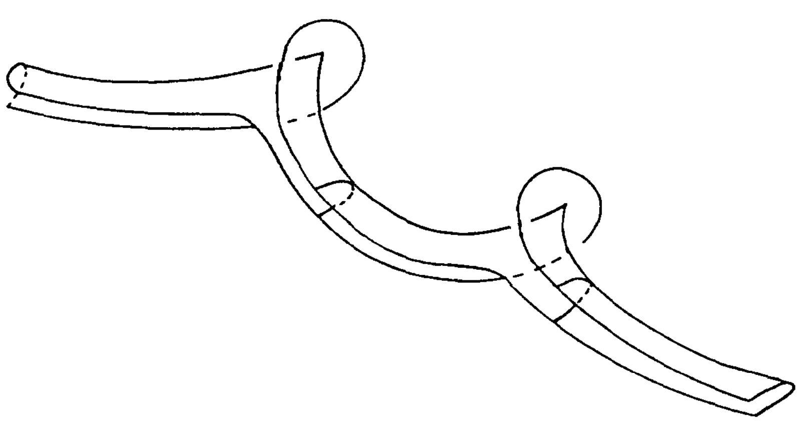

Fig. 9. Image of a fold curve.

restriction of $f$. The extension to the neighborhood of $C_f$ may be completed to $M$ by smoothly extending the height function to all of $M$.

Theorem 2 follows from Lemma 2 and the observation that ker $df|_C$ is orientable if and only if $C$'s with cylinder neighborhoods have an even number of cusps and $C$'s with Möbius neighborhoods have an odd number of cusps.

By way of conclusion of this section and before going on to some specific examples it is illuminating to consider the reason for the assumption by trying to lift a map of a cylinder neighborhood having a single cusp. This is illustrated in Fig. 10 where one observes, at the left of the picture the necessity of a configuration which is not allowed in an immersion, a pinch singularity.

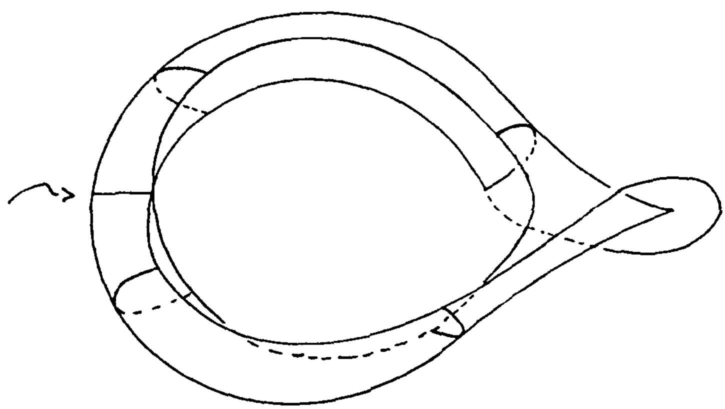

Fig. 10. Pinch singularity.

## 5. Several examples

Haefliger [2] has constructed several useful examples of generic maps of surfaces into $R^2$ which can be understood quite fruitfully by employing the methods of Section 2 and Section 3. To these we add an additional one, a generic map of $RP^2$ into $R^2$ having a single cusp and a connected fold curve.

First we shall consider Haefliger's example of a generic map, $f$, of a sphere into the plane which can not be the projection of an immersion into $R^3$. $C_f$ consists of two parallel fold curves, $C_1$ and $C_2$, each containing a single cusp. This is illustrated in Fig. 11 where the solid curves bound disks and the pair of dotted curves bound an annulus. If we begin to run these two cusps together we have the configuration in Fig. 12. After running the cusps together we do have a locus that can be achieved by the projection of an immersion as illustrated in Fig. 13.

The fold curve is precisely the Milnor curve, Poenaru [8] page 342–09, and the solid line is the boundary of one of the inequivalent immersed disks as pictured in Fig. 14 while the dotted line is the boundary of the other immersed disk as pictured in Fig. 15. Their union gives an immersed sphere in $R^3$ whose fold curve projects

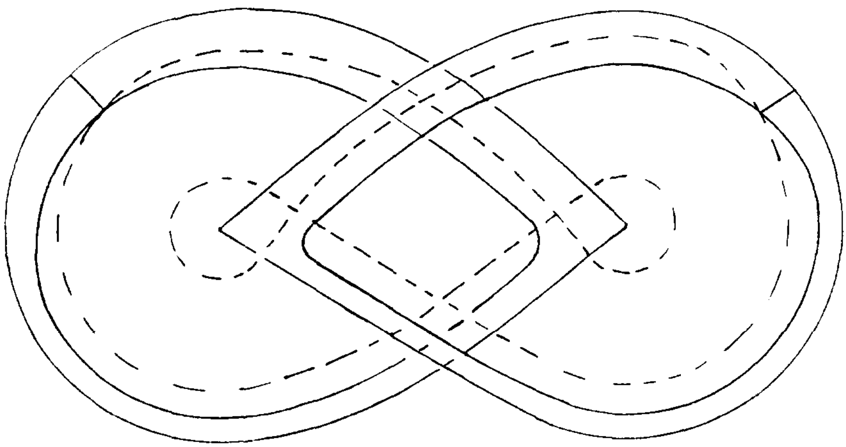

Fig. 11.

to that shown in Fig. 13. The change between that immersed surface and the map having the fold curves pictured in Fig. 12 occurs in the critical region of Fig. 13 and is pictured here in Figs. 16 and 17. The fundamental local reason for the failure to achieve an immersion is the necessity of a pinch singularity as illustrated in Fig. 18. Such crossings must always occur in pairs if the domain is orientable and can, therefore, be removed by reversing the process illustrated in Figs. 16 and 17.

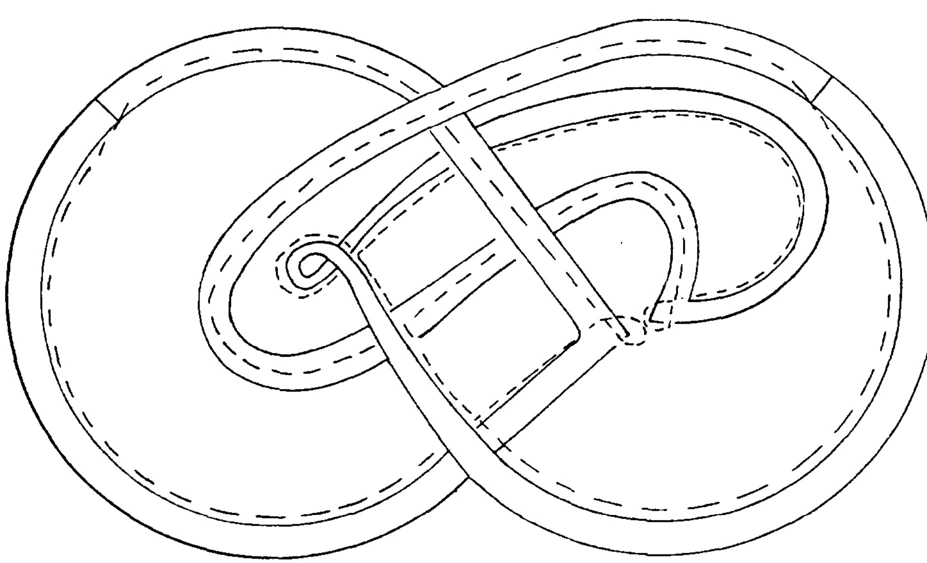

Fig. 12.

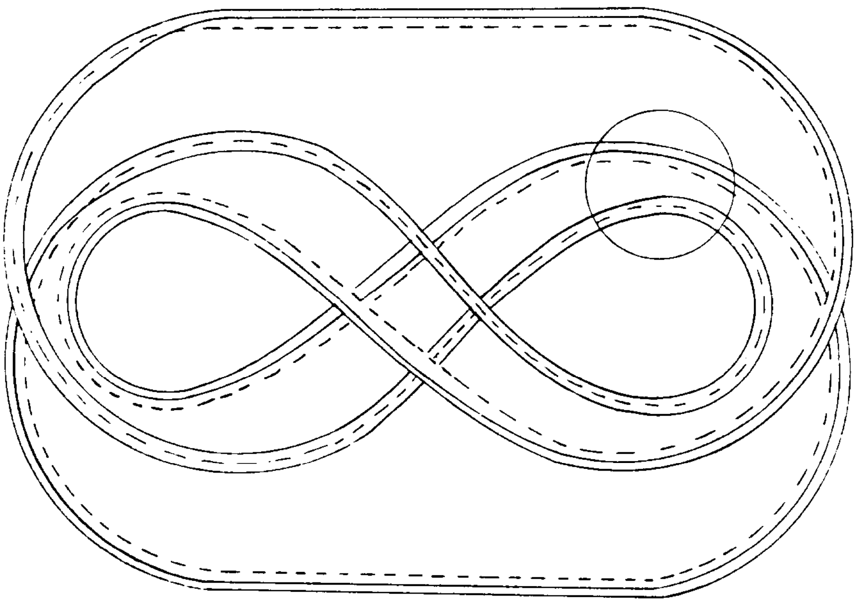

Fig. 13.

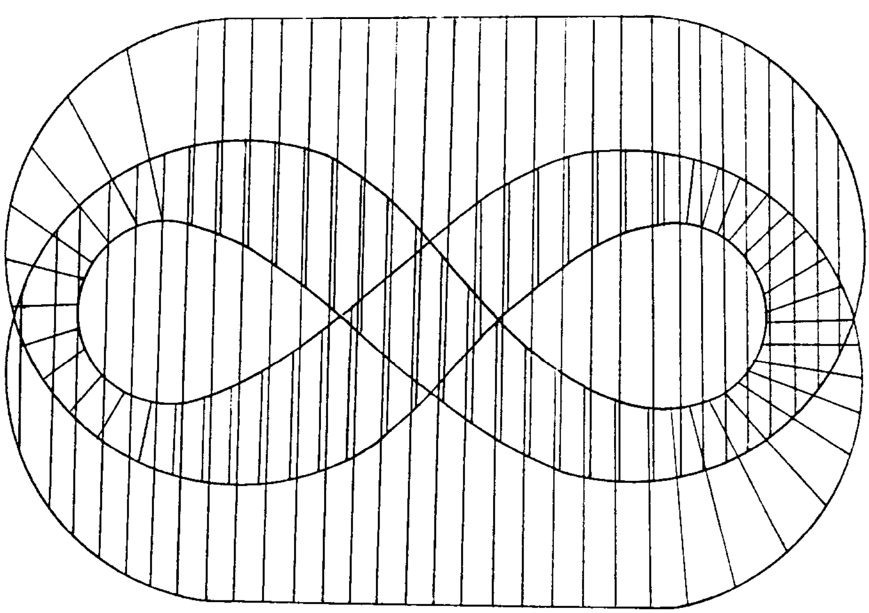

Fig. 14.

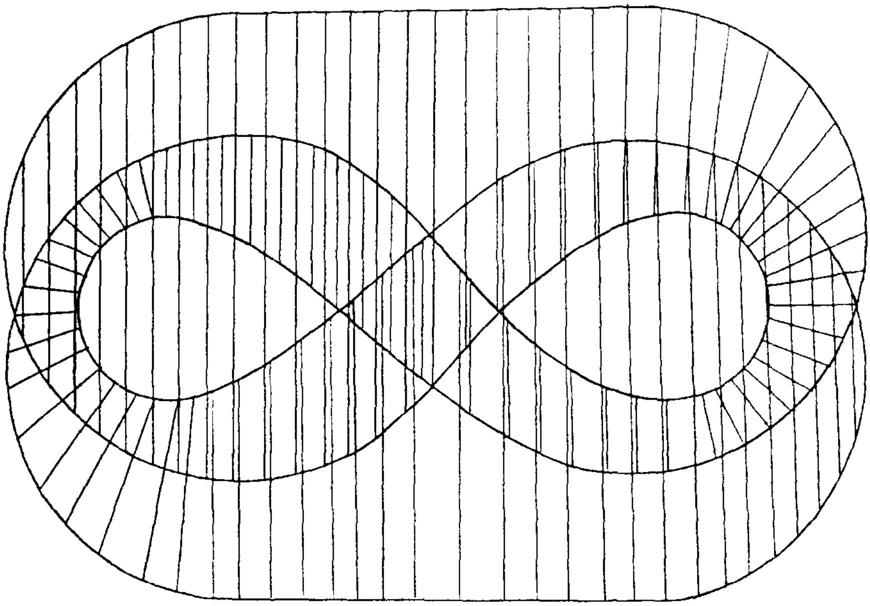

Fig. 15.

Finally we describe an immersion of $RP^2$ into $R^3$ whose projection to $R^2$ has a connected fold curve which contains a single cusp. As with the $S^2$ case we will first describe the fold curve, in Fig. 19, and then describe the immersion of the disk which is attached to the boundary of the Möbius band neighborhood of the fold curve. The boundary of the immersed disk is shown in Fig. 20 and various stages of the immersion are illustrated in Figs. 21 through 24. Note the remarkable similarity of the construction of the immersions of the sphere and $RP^2$ and the Milnor curve.

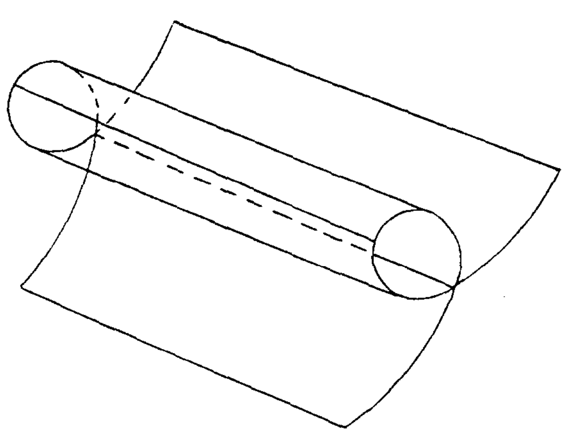

Fig. 16. Immersion of $S^2$.

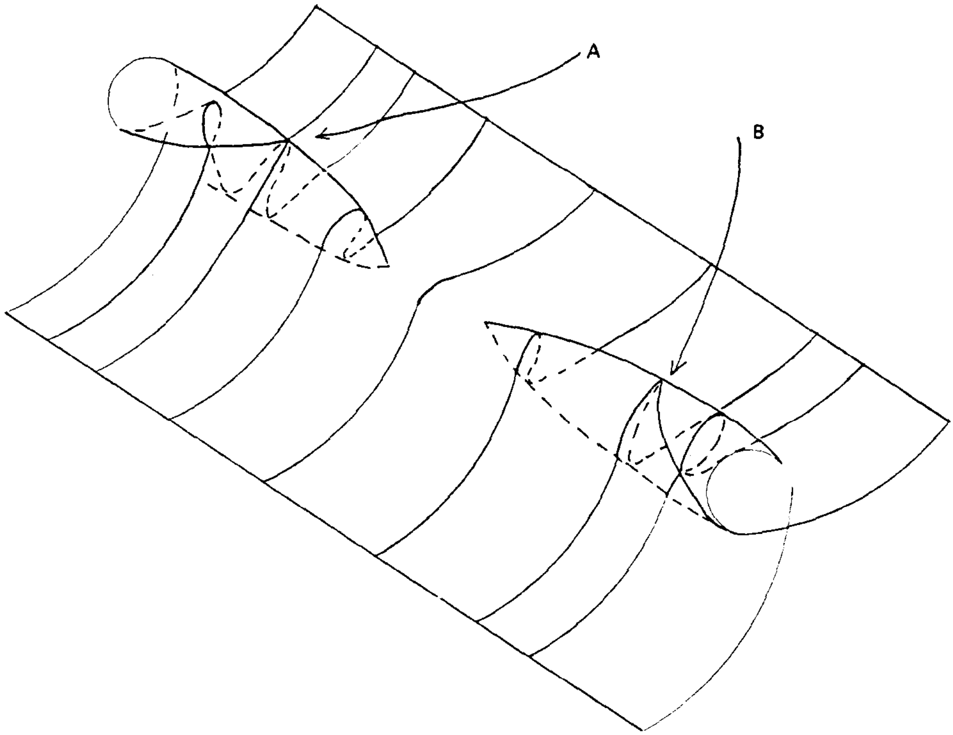


Fig. 17. No immersion exists due to the pinch points $A$ and $B$.

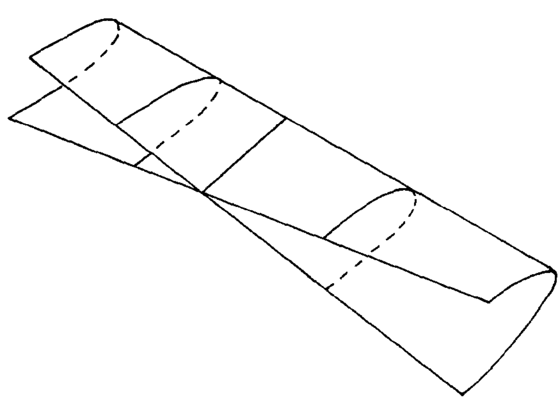

Fig. 18. Pinch singularity.

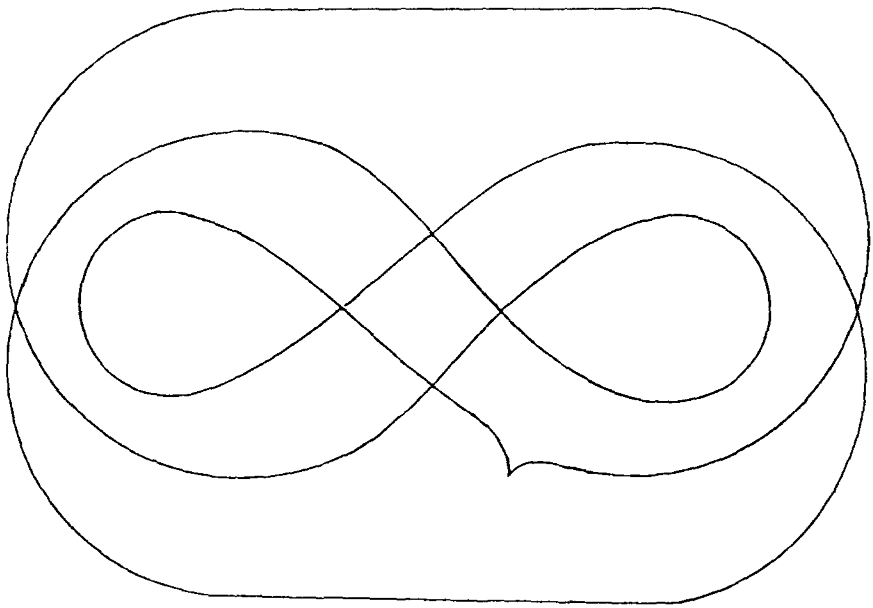

Fig. 19. $RP^2$ Fold curve.

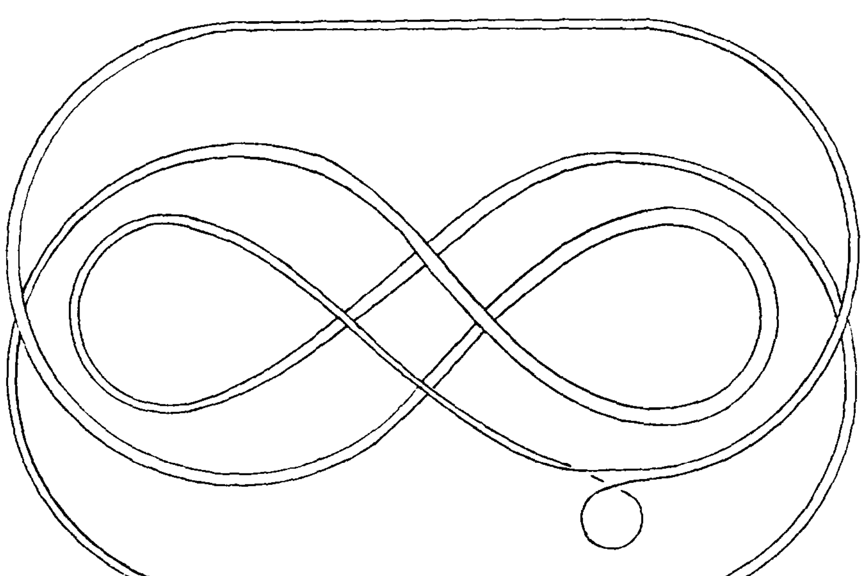

Fig. 20. Boundary of immersed disk.

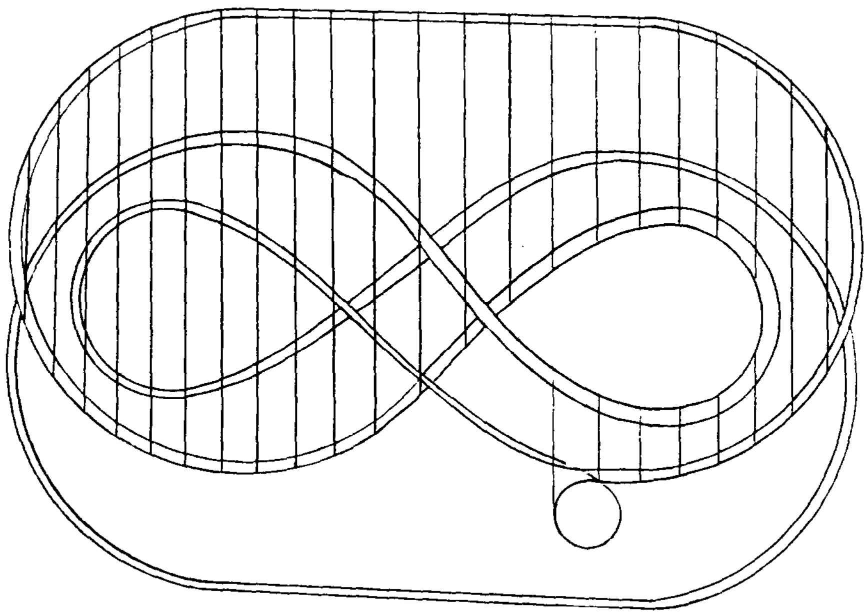

Fig. 21. Immersion of disk (1).

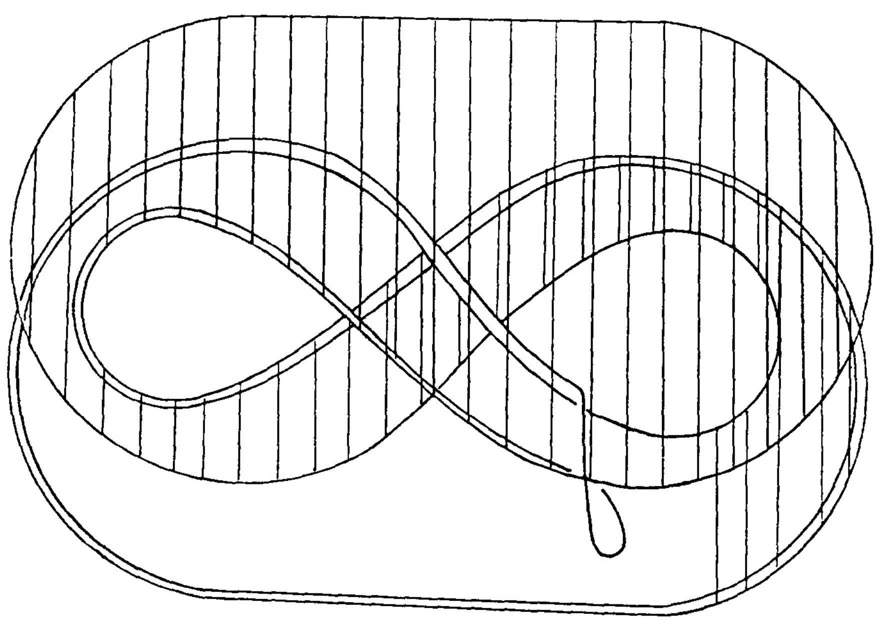

Fig. 22. Immersion of disk (2).

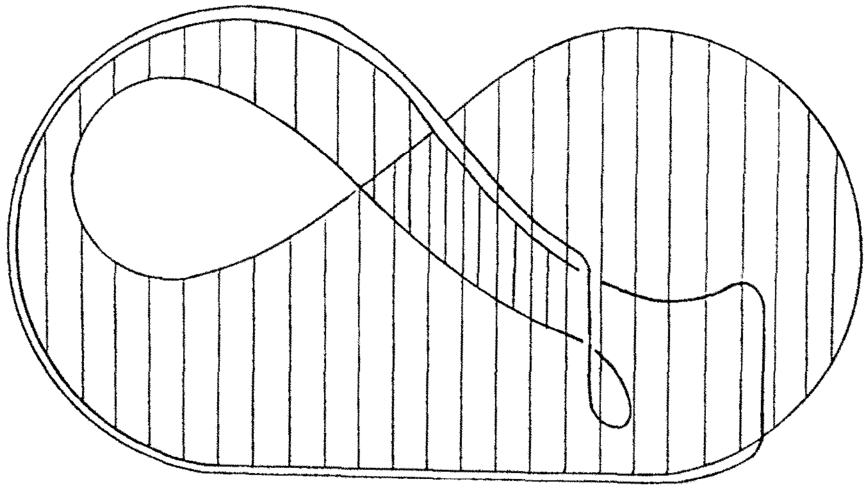

Fig. 23. Immersion of disk (3).

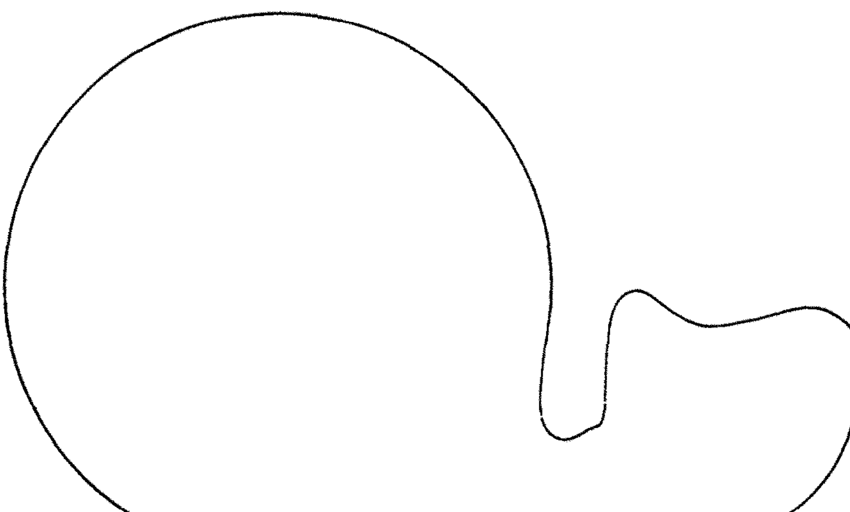

Fig. 24. Immersion of disk (4).

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